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THE EFFECT OF VIBRATION ON QUARTZ CRYSTAL RESONATORS.(U)
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RESEARCH AND DEVELOPMENT TECHNICAL REPORT

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# THE EFFECT OF VIBRATION ON QUARTZ CRYSTAL RESONATORS

Raymond L. Filler

ELECTRONICS TECHNOLOGY & DEVICES LABORATORY

May 1980



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The acceleration sensitivity of crystal oscillators is assuming increasing significance in modern communications, navigation and identification systems. This report describes how acceleration, particularly vibration, effects the frequency of quartz crystal controlled oscillators. Simple harmonic motion, complex and random vibrations, and frequency multiplication are covered. Several examples and experimental demonstrations are given.

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## INTRODUCTION

This report is a tutorial for readers interested in the effects of vibration on the output frequency of a quartz crystal controlled oscillator. It is a compendium of information gathered from various published sources plus several experimental examples. Individuals seeking more details are encouraged to refer to the works cited in the references.

The acceleration sensitivity of crystal oscillators is assuming increasing significance in modern communications, navigation and identification systems. The requirements that these systems be secure and resistant to jamming imposes stringent requirements on the frequency control aspects of the system designs. For example, in systems such as the NAVSTAR Global Positioning System (GPS), the vibration sensitivity of the crystal oscillator is a major limitation on the jam resistance achievable in a mobile (i.e., vibratory) environment.[1,2]

The resonant frequency of a quartz crystal resonator is influenced by acceleration.[3] A classic example of this is the static "2g tip-over" test, whereby a resonator exhibits a variation in resonant frequency depending on its orientation with respect to the earth's gravitational field. The rotation of a resonator by $180^{\circ}$ about an axis parallel to the surface of the earth causes a reversal of sign and therefore a change in the gravitational acceleration vector of amplitude 2g (g being the acceleration due to gravity at sea level). The source of the coupling between the frequency and the gravitational field is most probably the stress sensitivity of quartz.[4] A stress applied to a quartz plate will change the resonant frequency of that plate. The effect is linear for small stresses.[5] A variation in the acceleration direction will vary the stress distribution and therefore the frequency. The "2g tip-over" test demonstrates that a change in sign of the acceleration vector with respect to the resonator will change the sign of the frequency shift.

The acceleration due to gravity (even of a static resonator) is indistinguishable from an inertial acceleration.[6] Therefore, a resonator undergoing inertial acceleration will exhibit a frequency shift with a magnitude dependent on the acceleration magnitude and sign dependent on the direction of the acceleration.

## SIMPLE HARMONIC MOTION

Consider a resonator (presumably in some sort of oscillator circuit so that one can monitor the resonant frequency) undergoing simple harmonic motion (i.e., a simple one frequency vibration). The position of the resonator as a function of time may be written as

$$X(t) = (\Delta X_{max}/2) \cos(2\pi f_v t) \qquad (1)$$

where

$X(t)$ = position as a function of time

$$f_v = \text{vibration frequency}$$

$$\Delta X_{max} = \text{peak-to-peak displacement.}$$

Let A(t) be the acceleration as a function of time. Since

$$A(t) = d^2X(t)/dt^2 \tag{2}$$

we get

$$A(t) = -(2\pi f_v t)^2 \; (\Delta X_{max}/2) \cos (2\pi f_v t) \tag{3}$$

or

$$A(t) = -(A_{max}) \cos (2\pi f_v t) \tag{3a}$$

where

$$A_{max} = (2\pi f_v t)^2 \; (\Delta X_{max}/2) \tag{3b}$$

If a resonator experiences such a motion, the frequency, $F_O$, of the output signal will be modulated in phase with the acceleration, due to the time varying stress, by an amount proportional to the acceleration magnitude. The frequency excursion, $\Delta F(t)$, is then

$$\Delta F(t) \propto A(t)F_o \tag{4}$$

where the constant of proportionality is known as the vibration sensitivity, which will be denoted by $\gamma$. The maximum frequency excursion is then related to the acceleration by the following relation:

$$\Delta F_{max} = A_{max}\gamma F_o. \tag{5}$$

The output frequency is, therefore,

$$F(t) = F_o + \Delta F_{max} \cos(2\pi f_v t) \tag{6}$$

where

$$F(t) = \text{output frequency}$$

$$F_o = \text{"at rest" frequency}$$

Figure 1 is a representation of the output frequency, $F(t)$, at different times. It can be seen that the instantaneous value of $F(t)$ is modulated by an amount $\Delta F$ about $F_o$ at a rate of $f_v$. The system designer must take this modulation into account. A narrow bandwidth servo-loop can lose lock if the frequency deviates too far from center.

The output voltage of the monitoring oscillator is

$$V(t) = V_o \cos \Phi(t) \tag{7}$$

where $\Phi(t)$, the "phase", is related to the frequency of the resonator by

$$\Phi(t) = 2\pi \int_o^t F(t)\, dt \tag{8}$$

therefore

$$\Phi(t) = 2\pi \int_o^t F_o\, dt + 2\pi \int_o^t \Delta F_{max} \cos(2\pi f_v t)\, dt \tag{9}$$

and

$$\Phi(t) = 2\pi F_o t + (\Delta F_{max}/f_v) \sin(2\pi f_v t) \tag{10}$$

whence it follows

$$V(t) = V_o \cos(2\pi F_o t + (\Delta F_{max}/f_v) \sin(2\pi f_v t)) \tag{11}$$

Figure 2 is an example of the output voltage of a frequency modulated resonator. The low frequency curve is the modulating frequency (vibration).

It is very difficult to quantify data in the form displayed in Figure 2. This representation is known as the time domain representation (the horizontal axis is time). A much clearer picture can be gained by

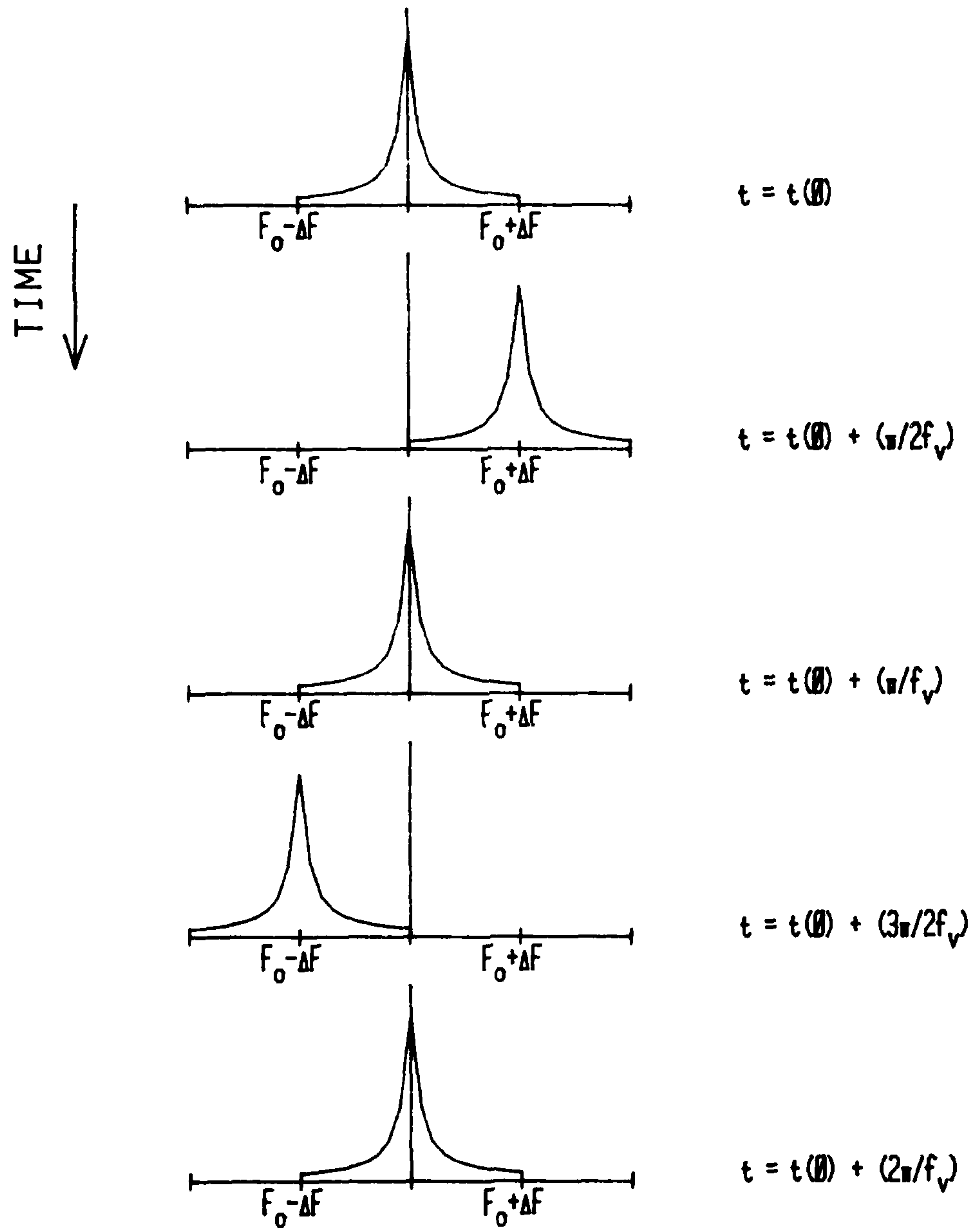


FIGURE 1 - THE 'INSTANTANEOUS' OUTPUT FREQUENCY OF A FREQUENCY MODULATED CARRIER

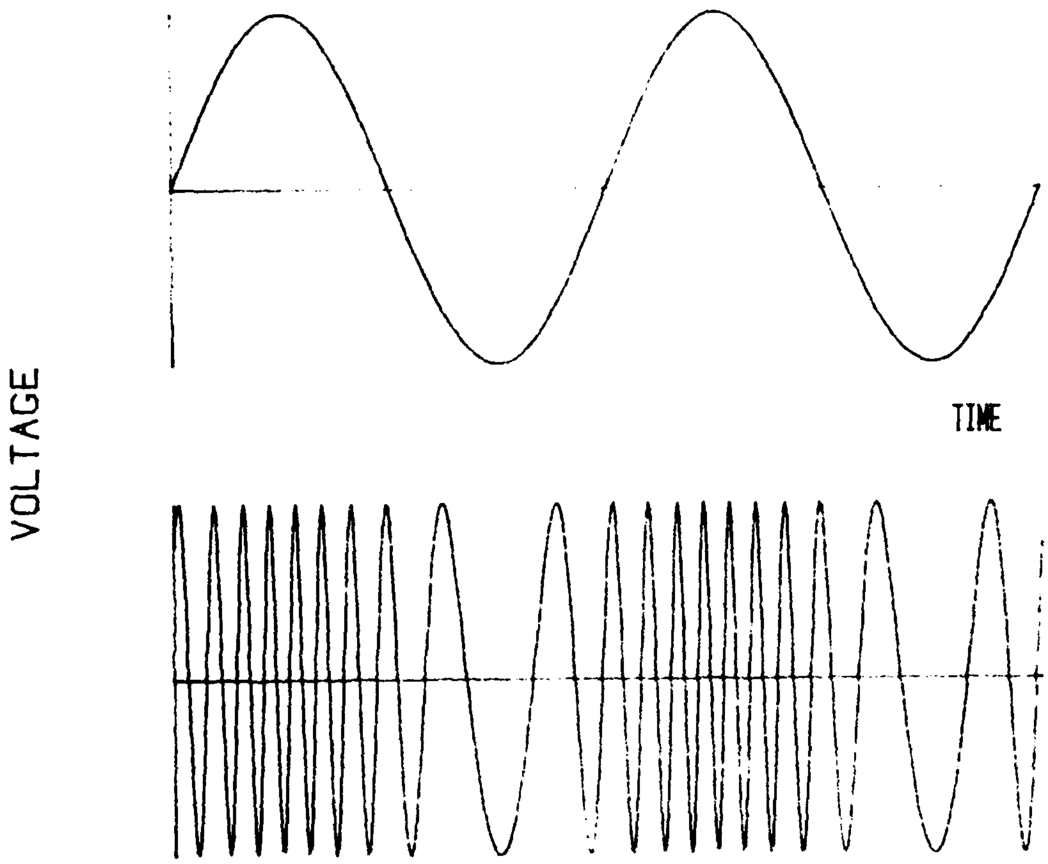


FIGURE 2 - THE OUTPUT VOLTAGE OF A FREQUENCY MODULATED CARRIER AND THE MODULATING SIGNAL

presenting the same information in the frequency domain where the output voltage of the oscillator is characterized as a function of frequency. To measure the output of an oscillator in the frequency domain, one must measure the signal passed through a series of narrowband transmission filters. One can perform this task most conveniently with a spectrum analyzer. In mathematical terms the frequency domain output of the spectrum analyzer corresponds to the decomposition of a complex waveform into its frequency (Fourier) components.

FREQUENCY DOMAIN

The decomposition of V(t) into its frequency components is most simply performed with the aid of several trigonometric identities.[7]

$$\cos(X + Y) \equiv \cos(X)\cos(Y) - \sin(X)\sin(Y) \tag{12}$$

$$\cos(X)\cos(Y) \equiv \tfrac{1}{2}\{\cos(X + Y) + \cos(X - Y)\} \tag{13}$$

$$-\sin(X)\sin(Y) \equiv \tfrac{1}{2}\{\cos(X + Y) - \cos(X - Y)\} \tag{14}$$

and two relations derived in Appendix I

$$\cos(\beta \sin X) = J_o(\beta) + 2\sum_{n=1}^{\infty} J_{2n}(\beta)\cos(2nx) \tag{15}$$

$$\sin(\beta \sin X) = 2\sum_{n=o}^{\infty} J_{2n+1}(\beta)\sin((2n+1)x) \tag{16}$$

where $Jp(\beta)$ = Bessel function of order p (see Appendix 1) and argument $\beta$. From equation 11 above

$$V(t) = V_o\{\cos(2\pi F_o t + (\Delta F_{max}/f_v)\sin(2\pi f_v t))\} \tag{17}$$

using equation 12 with $X = 2\pi F_o t$ and $Y = (\Delta F_{max}/f_v)\sin(2\pi f_v t)$ (18)
we get

$$V(t) = V_o\{\cos(2\pi F_o t)\cos((\Delta F_{max}/f_v)\sin(2\pi f_v t))$$

$$-\sin(2\pi F_o t)\sin((\Delta F_{max}/f_v)\sin(2\pi f_v t))\} \tag{19}$$

using equations 15 and 16 with $\beta = (\Delta F_{max}/f_v)$ and $X = 2\pi f_v t$ we get

$$V(t) = V_o\{\cos(2\pi F_o t)(J_o(\Delta F_{max}/f_v)$$

$$+ 2 \sum_{n=1}^{\infty} J_{2n}(\Delta F_{max}/f_v) \cos(4n\pi f_v t))$$

$$-\sin(2\pi F_o t)(2 \sum_{n=o}^{\infty} J_{2n+1}(\Delta F_{max}/f_v) \sin((2n+1) 2\pi f_v t))\} \quad (20)$$

which reduces to

$$V(t) = V_o\{J_o(\Delta F_{max}/f_v) \cos(2\pi F_o t)$$

$$+ 2 \sum_{n=1}^{\infty} J_{2n}(\Delta F_{max}/f_v) \cos(2\pi F_o t) \cos(4\pi n f_v t)$$

$$- 2 \sum_{n=o}^{\infty} J_{2n+1}(\Delta F_{max}/f_v) \sin(2\pi F_o t) \sin((2n+1) 2\pi f_v t)\} \quad (21)$$

using equation 13 with $X = 2\pi F_o t$ and $Y = 4n\pi f_v t$ and equation 14 with $X = 2\pi F_o t$ and $Y = (2n+1) 2\pi f_v t$ we get

$$V(t) = V_o\{J_o(\Delta F_{max}/f_v) \cos(2\pi F_o t)$$

$$+ \sum_{n=1}^{\infty} J_{2n}(\Delta F_{max}/f_v)(\cos(2\pi(F_o+2nf_v)t) + \cos(2\pi(F_o-2nf_v)t)$$

$$+ \sum_{n=o}^{\infty} J_{2n+1}(\Delta F_{max}/f_v)(\cos(2\pi(F_o t(2n+1)f_v)t)$$

$$- \cos(2\pi(F_o-(2n+1))f_v t\} \quad (22)$$

which expands to

$$V(t) = V_o\{J_o(\Delta F_{max}/f_v) \cos(2\pi f_v t) + J_1(\Delta F_{max}/f_v) \cos(2\pi(F_o+f_v)t)$$

$$- J_1(\Delta F_{max}/f_v) \cos(2\pi(F_o-f_v)t) + J_2(\Delta F/f_v) \cos(2\pi(F_o+2f_v)t)$$

$$+ J_2(\Delta F_{max}/f_v) \cos(2\pi(F_o-2f_v)t) + \ldots .\} \quad (23)$$

So the spectrum of a sinusoidally vibrated resonator contains an infinite number of Fourier components on each side of the carrier, separated from the carrier by $f_v$, $2f_v$, $3f_v$ etc. Each of these terms

is called a vibration induced sideband.

Example 1:

The acceleration sensitivity of AT-cut quartz is typically about $2 \times 10^{-9}/g$. For a resonator with a frequency of 5 MHz, undergoing a harmonic vibration with maximum acceleration of 5g, the $\Delta F$ is

$$\Delta F_{max} = \gamma A_{max} F_o = (2 \times 10^{-9}/g)(5g)(5 \times 10^6 \text{ Hz}) \tag{24}$$

$$\Delta F_{max} = 0.05 \text{ Hz}$$

If the vibration frequency is above 5 Hz, the value of $\Delta F_{max}/f_v$ is below 0.01. As can be seen in Appendix II, for values of $\Delta F_{max}/f_v$ <0.1.

$$J_o(\Delta F_{max}/f_v < 0.1) \simeq 1$$

$$J_o(\Delta F_{max}/f_v < 0.1) \simeq (\Delta F_{max}/f_v)/2 \tag{25}$$

$$J_{p>1}(\Delta F_{max}/f_v < 0.1) \simeq 0$$

Therefore, only the first upper and lower sidebands are important for small modulations. Mathematically, this reduces equation 23 to

$$V(t) \simeq V_o\{J_o(\Delta F_{max}/f_v)\cos(2\pi F_o t) + J_1(\Delta F_{max}/f_v)\cos(2\pi(F_o+f_v)t)$$

$$- J_1(\ F_{max}/f_v)\cos(2\ (F_o-f_v)t)\} \tag{26}$$

Now let $\mathcal{L}_v^n(f_v)$ be defined as the ratio of the power contained in the $n^{th}$ vibration induced sideband to the carrier power. That is

$$\mathcal{L}_v^1(f_v) = \{J_1(\Delta F_{max}/f_v)/J_o(\Delta F_{max}/f_v)\}^2 \tag{27}$$

to express $\mathcal{L}$ in decibels

$$\mathcal{L}_v^n(f_v)(\text{dB}) = 10 \log \mathcal{L}_v^n(f_v) \qquad \text{or} \tag{28}$$

$$\mathcal{L}_v^n(f_v)(\text{dB}) = 20 \log\{J_1(\Delta F_{max}/f_v)/J_o(\Delta F_{max}/f_v)\} \tag{29}$$

For $\Delta F_{max}/f_v < 0.1$

$$\mathcal{L}_v^n(f_v)(dB) = 20 \log (\Delta F_{max}/2f_v) \tag{30}$$

For our current example:

$$\begin{aligned}
\mathcal{L}_v^1 &(1 \text{ Hz}) = -32\text{dB} \\
\mathcal{L}_v^1 &(5 \text{ Hz}) = -46\text{dB} \\
\mathcal{L}_v^1 &(25 \text{ Hz}) = -60\text{dB} \\
\mathcal{L}_v^1 &(50 \text{ Hz}) = -66\text{dB} \\
\mathcal{L}_v^1 &(500 \text{ Hz}) = -86\text{dB} \\
\mathcal{L}_v^1 &(5000 \text{ Hz}) = -106\text{dB}
\end{aligned} \tag{31}$$

The negative sign means that the sideband is smaller than the carrier. In the 25 Hz case, for example, the sideband is smaller by a factor of $1 \times 10^6$ in power, or a 1 volt carrier signal will have a 1 millivolt sideband.

It is seen that the lower the vibration frequency the smaller the difference between the carrier and the sideband amplitudes, i.e., the signal to noise ratio is degraded. Unfortunately, the low frequency vibrations are the ones most difficult to damp.

Figure 3 is an actual trace of the power spectrum of a resonator at rest. The zero frequency point on the abcissa is the resonant frequency, $F_o$. The abcissa is the Fourier frequency minus the oscillator "at rest" frequency.

Figure 4 is the power spectrum of the same resonator being vibrated at a level of 10g at a vibration frequency of 100 Hz. The lobes on either side of the carrier are the vibration induced sidebands. The ordinate is in units of dB below the carrier. It can be seen that the ratio of the power in the sideband to that of the carrier is $-60.1$ dB. This translates into an acceleration sensitivity of $4.0 \times 10^{-9}$/g.

FREQUENCY MULTIPLICATION

In systems use, it is common for an oscillator output frequency to be multiplied to some higher frequency. Consider a vibration modulated frequency, as in the previous section.

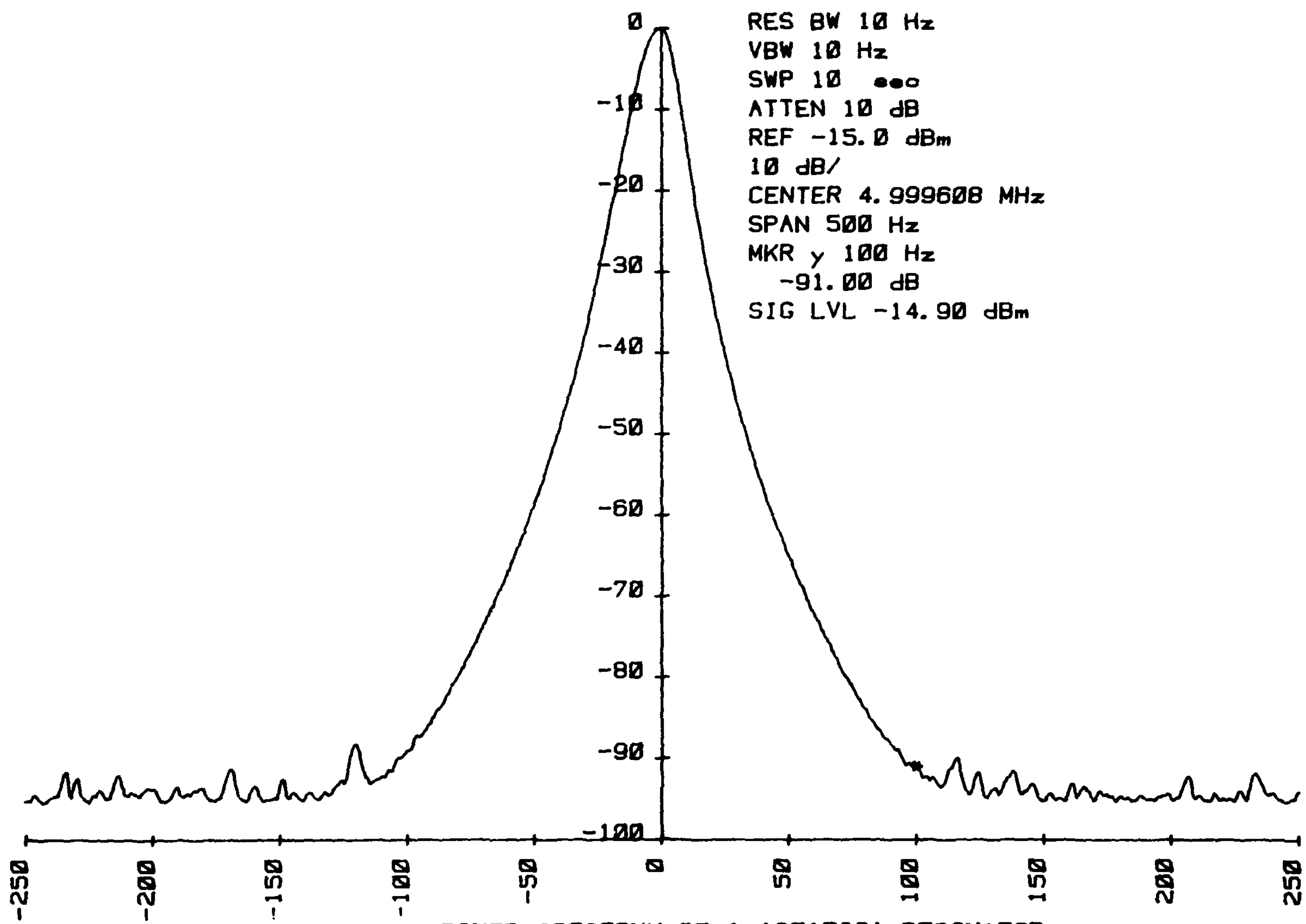


FIGURE 3 - POWER SPECTRUM OF A 'STATIC' RESONATOR

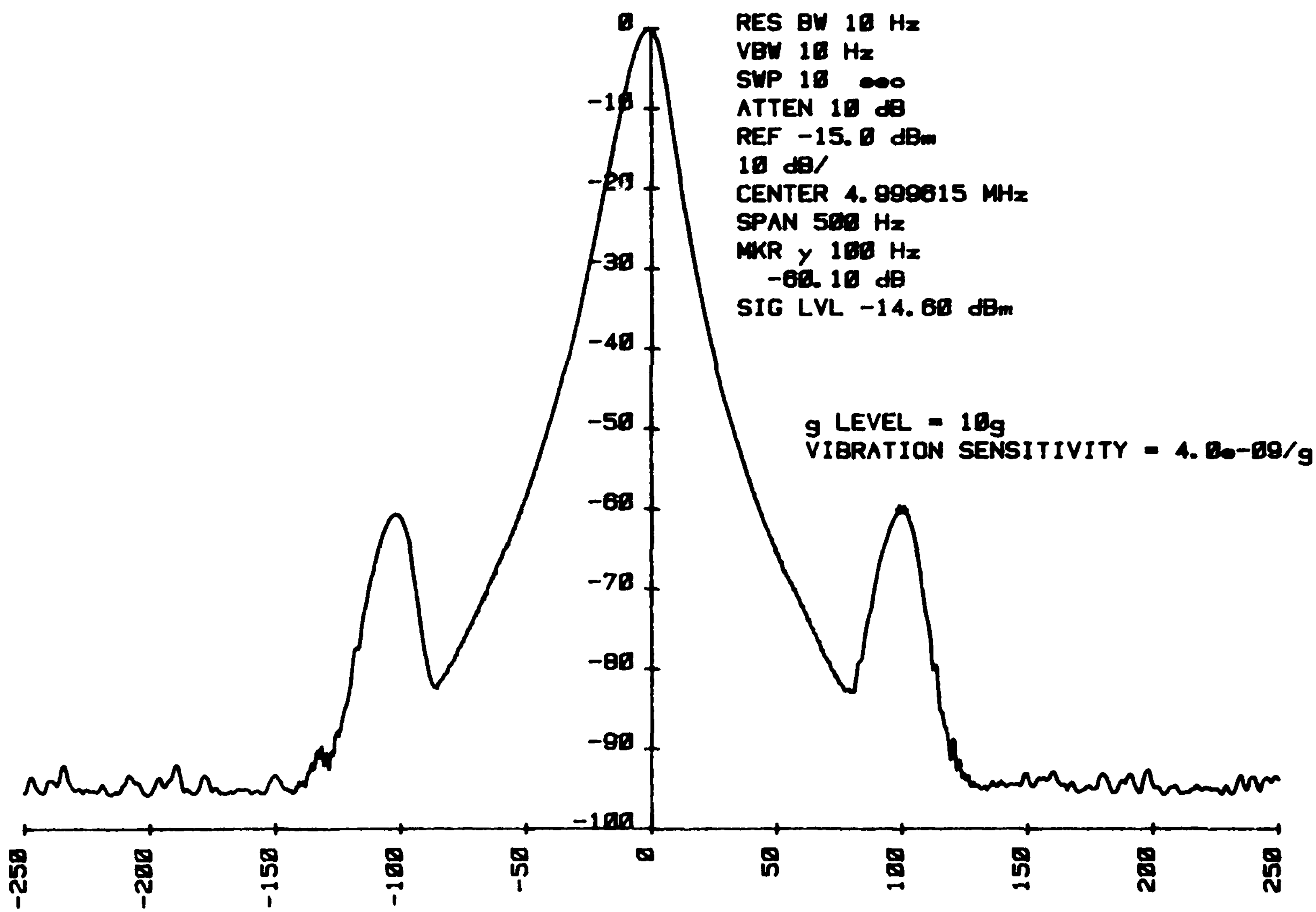


FIGURE 4 - POWER SPECTRUM OF A RESONATOR UNDERGOING 100Hz, 10g VIBRATION

$$F(t) = F_o + \Delta F_{max} \cos(2\pi f_v t) \tag{32}$$

If this frequency is multiplied by a factor, N, the new frequency, $F_m(t)$, is then

$$F_m(t) = NF(t) = N(F_o + \Delta F_{max} \cos(2\pi f_v t)) \tag{33}$$

As earlier, the phase $\Phi_m(t)$ is given by

$$\Phi_m(t) = 2\pi \int_0^t F_m(t)dt = 2\pi N \int_0^t F(t)dt \tag{34}$$

We have also seen that this may be written as

$$\Phi_m(t) = 2\pi NF_o t + (N\Delta F_{max}/f_v) \sin(2\pi f_v t) \tag{35}$$

so $$V(t) = V_o \cos(2\pi NF_o t + (N\Delta F_{max}/f_v) \sin(2\pi f_v t)). \tag{36}$$

From this it follows that

$$V(t) = V_o\{J_o(N\Delta F_{max}/f_v) \cos(2\pi NF_o t) + J_1(N\Delta F/f_v) \cos(2\pi(NF_o t + f_v)t)$$

$$-J_1(N\Delta F_{max}/f_v) \cos(2\pi NF_o - f_v)t) + \ldots\} \tag{37}$$

Therefore, for a frequency multiplied signal

$$\mathcal{L}_v^1(f_v) = 20 \log\{J_1(N\Delta F_{max}/f_v)/J_o(N\Delta F_{max}/f_v)\} \tag{38}$$

One must be careful not to use inappropriate approximations for Jp(X) if X > 0.1.

To illustrate the effects of frequency multiplication, we shall choose an example from the GPS system.

Example 1: Let
$$\begin{aligned} F_o &= 5.115 \text{ MHz} \\ \gamma &= 2 \times 10^{-9}/g \\ A_{max} &= 5g \\ f_v &= 52 \text{ Hz} \end{aligned} \tag{39}$$

Then $\mathcal{L}_v^1(52 \text{ Hz}) = 20 \log (A_{max}\gamma F_o/2f_v) = 20 \log (4.9 \times 10^{-4})$

$$\mathcal{L}_v^1(52 \text{ Hz}) = -66\text{dB}$$

If this signal is multiplied by N = 308 to get to 1575 MHz, we get

$$\mathcal{L}_v^1(52 \text{ Hz}) = 20 \log\{J_1(NF_oA_{max}\gamma/f_v)/J_o(NF_oA_{max}\gamma/f_v)\} \tag{40}$$

since

$$NF_oA_{max}\gamma/f_v = 0.31 \tag{41}$$

we must use the full values of $J_o$ and $J_1$ from the appendix.

$$\mathcal{L}_v^1(52 \text{ Hz}) = 20 \log (.1483/.997) = -16\text{dB} \tag{42}$$

The sideband is thus only 16dB below the carrier.

Example 2:

If we lower the vibration frequency to 10 Hz, we get

$$\mathcal{L}_v^1(10 \text{ Hz}) = 20 \log\{J_1(1.51)/J_o(1.51)\} = +0.87\text{dB} \tag{43}$$

The plus sign denotes that the sideband is actually larger than the carrier!

Example 3:

If $f_v$ = 6.55 Hz with $A_{max}$ still 5g and N = 308.

$$\mathcal{L}_v^1(6.55 \text{ Hz}) = 20 \log\{J_1(2.405)/J_o(2.405)\} \tag{44}$$

Since $J_o(2.405) = 0$, $\mathcal{L}_v^1(6.55 \text{ Hz}) = \infty$, i.e., the carrier disappears!!

Figure 5, an experimental demonstration, is a comparison of the power spectrum of a resonator being vibrated with its output frequency multiplied by a factor of 10, and the same resonator with an unmultiplied

output. Since

$$\mathcal{L}_v^{\,1}(f_v)(N = 1) = 20\ \log(A_{max}F_o\gamma/2f_v) \tag{45}$$

and

$$\mathcal{L}_v^{\,1}(f_v)(N = 10) = 20\ \log(10\ A_{max}F_o\gamma/2f_v) \tag{46}$$

$$\mathcal{L}_v^{\,1}(N = 10) - \mathcal{L}_v^{\,1}(N = 1) = 20\ \log(10\ A_{max}F_o\gamma/2f_v)/(A_{max}F_o\gamma/2f_v)$$

$$\mathcal{L}_v^{\,1}(N = 10) - \mathcal{L}_v^{\,1}(N = 1) = 20\ \log(10) = 20\text{dB}$$

in Figure 5 $\mathcal{L}(N_1 = 10) = -42$dB and $\mathcal{L}(N = 1) = -62$dB so $\mathcal{L}_v^{\,1}(N = 10) - \mathcal{L}_v^{\,1}(N = 1) = 20$dB. It can be seen that as one increases the multiplicative factor, N, or decreases the vibration frequency, $f_v$, the modulation factor, $(N\Delta F_{max}/f_v)$, will increase. The oscillatory behavior of the Bessel Functions leads to the growing importance of higher order sidebands.

Example 4: As before, if

$$\begin{aligned} F_o &= 5.115\ \text{MHz} \\ \gamma &= 2 \times 10^{-9}/g \\ A_{max} &= 5g \\ N &= 308 \\ f_v &= 52\ \text{Hz} \end{aligned} \tag{47}$$

$$\mathcal{L}_v^{\,1}(52) = -16\text{dB}$$

but $$\mathcal{L}_v^{\,2}(52) = -38\text{dB}$$

The second order sideband is only 20dB below the first order sideband. Before multiplication the second order sideband was 138dB below the carrier and 72dB below the first order sideband. When $f_v = 6.55$ Hz, the carrier disappears and the second order sideband is only 1.6dB below the first order sideband.

It should be noted that it is not the introduction of the multiplicative factor which causes the redistribution of the power from the carrier to the sidebands, but rather the increase in absolute frequency. $\mathcal{L}$ is a function of $(N\Delta F_{max}/f_v)$ which can be written as $(\gamma A_{max}\ NF_o/f_v)$.

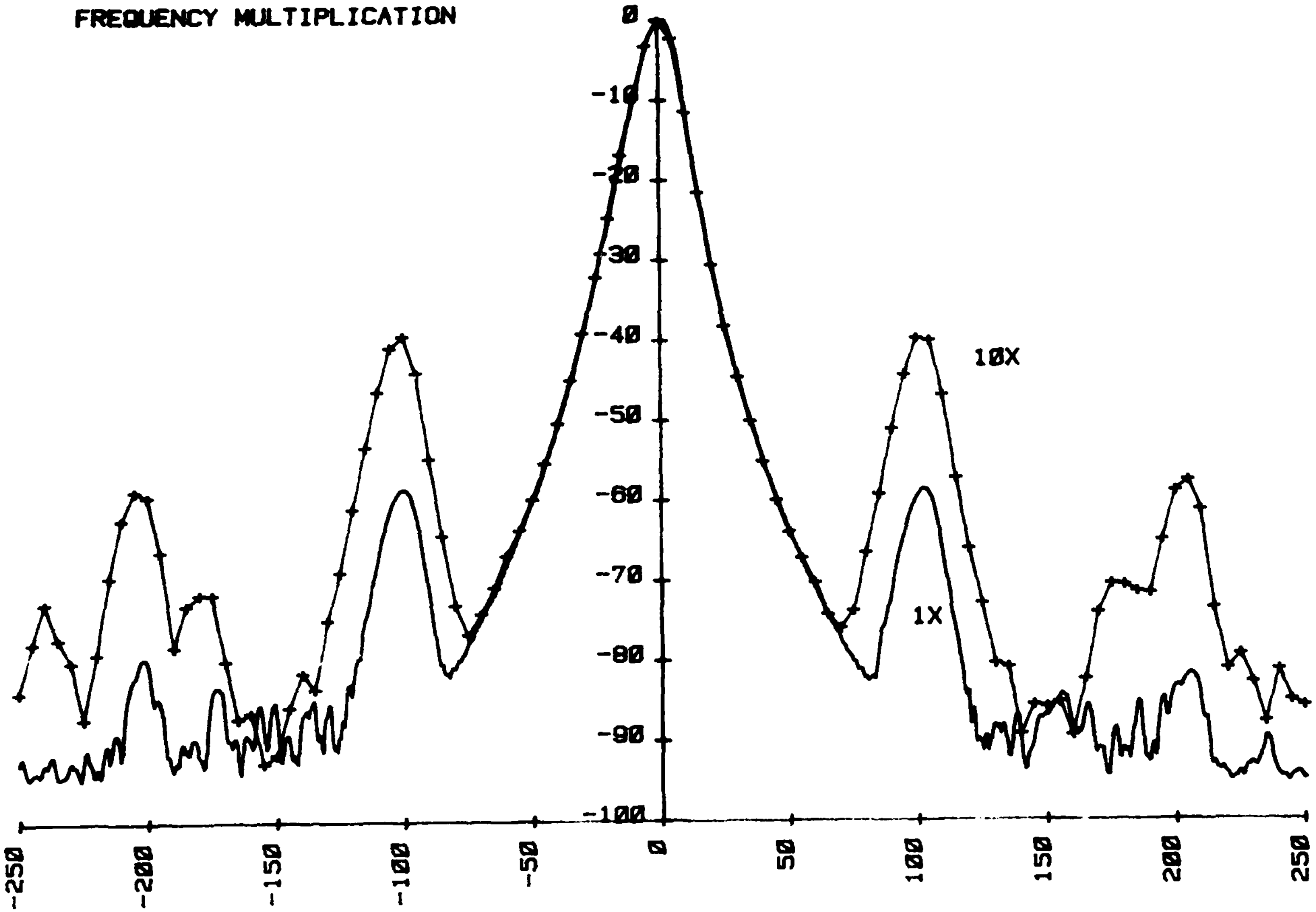


FIGURE 5 - THE RESULT OF FREQUENCY MULTIPLICATION

If $NF_o$ is relabeled as F, $\mathcal{L}$ is then a function of $(\gamma A_{max} F/f_v)$. It does not matter whether F is obtained by $F = F_o$ directly and $N = 1$ or by $F_o$ being multiplied up to F. As a matter of fact, a fractional value for N exhibits the same vibration response. The only way to reduce the absolute frequency excursion, i.e., improve the vibration insensitivity, is to reduce the factor $\gamma$.

Example 5:

A vibration resistant oscillator is needed at 1575 MHz. The designer has the option of using a 5.115 MHz bulk wave crystal oscillator multiplied up to 1575 MHz or using a 1575 MHz surface acoustic wave (SAW) oscillator. Both oscillators exhibit an acceleration sensitivity of $2 \times 10^{-9}$/g. Which oscillator should the designer choose?

In example 1 we calculated that $NF_o A_{max} \gamma / f_v = 0.31$ which implied that $\mathcal{L}_v^1(52\text{ Hz}) = -16$dB.

For the SAW oscillator, at the same vibration frequency $F_o A_{max}/f_v$ is again 0.31, which implies again that $\mathcal{L}_v^1(52\text{ Hz})$ is again $-16$dB.

Thus we see that the vibration induced sidebands are identical for the two cases. Therefore, the designer should base his decision on other factors, such as long term stability, or else choose the device that is available with a lower vibration sensitivity.

The result of example 6 is easier to understand in the time domain, e.g., as shown in equation 5 and Figure 1. The frequency of the 5.115 MHz oscillator will vary by $\gamma F_o A_{max} = 0.05$ Hz.

Upon multiplication by 308, both $F_o$ and $F_{max}$ increase by a factor of 308, to 1575 MHz and 15.75 Hz, respectively. The SAW oscillator frequency will vary about 1575 MHz by $\gamma F_o A_{max} = 15.75$ Hz, which is the same as the variation in the multiplied bulk wave oscillator frequency.

COMPLEX VIBRATION

If the resonator is being vibrated by a superposition of simple harmonic vibrations the situation may become more complicated. The frequency of the resonator becomes, in general:

$$F(t) = F_o + \sum_{n=1}^{\infty} \Delta F_{nmax} \cos(2\pi f_{vn} t) \qquad (48)$$

and the phase is therefore,

$$\Phi(t) = 2\pi F_o t + \sum_{n=1}^{\infty} (\Delta F_{max}/f_{vn}) \sin(2\pi f_{vn} t), \qquad (49)$$

therefore,

$$V(t) = V_o\{\cos(2\pi F_o t) + \sum_{n=1}^{\infty} (\Delta F_{max}/f_v) \sin(2\pi f_{vn} t)\} \tag{50}$$

which becomes

$$V(t) = V_o\{\cos(2\pi F_o t) \cos( \sum_{n=1}^{\infty} (\Delta F_{max}/f_v) \sin(2\pi f_{vn} t))$$

$$-\sin(2\pi F_o t) \sin( \sum_{n=1}^{\infty} (\Delta F_{max}/f_v) \sin(2\pi f_{vn} t)\}. \tag{51}$$

If, for simplicity, we choose only $2f_v$'s, we have

$$V(t) = V_o\{\cos(2\pi F_o t) \cos(\Delta F_{1max}/f_{v1}) \sin(2\pi f_v t) + (\Delta F_{2max}/f_{v2}) \sin(2\pi f_{v2} t))$$

$$- \sin(2\pi F_o t) \sin(\Delta F_{1max}/f_{v1}) \sin(2\pi f_{v1} t) + (\Delta F_2/f_{v2}) \sin(2\pi f_{v2} t))\} \tag{52}$$

This becomes unwieldly very quickly unless we make some simplifying assumptions. If we assume for the moment that $\Delta F_{max}/f_{vn}$ is small compared to 1 we may drop all terms with Bessel functions of order $\geq 2$ and all terms in $J_1{}^2$. V(t) becomes (see Appendix III)

$$V(t) = V\{J_o(\Delta F_{1max}/f_{v1}) J_o(\Delta F_{2max}/f_{v2}) \cos(2\pi F_o t)$$

$$-J_1(\Delta F_{1max}/f_{v1}) J_o(\Delta F_{2max}/f_{v2}) (\cos 2(\pi F_o - f_{v1})t) - \cos(2\pi(F_o + f_{v2})t)$$

$$- J_o(\Delta F_{2max}/f_{v2}) J_1(\Delta F_{max}/f_{v2}) (\cos(2\pi(F_o - f_{v2})t) - \cos(2\pi(F_o + f_{v2})t)) \tag{53}$$

For the approximation we have made this reduces to the linear sum of all the individual sidebands. For situations where $\Delta F_{nmax}/f_{v1} > 1$ the result is much more complex. In that case, less frequency modulation is required to mask the carrier.

## RANDOM VIBRATIONS

In this case the vibratory power is distributed over a range of frequencies. The worst case condition is described by a power spectral density plot of the vibration envelope.

The power spectral density of a vibration is the mean square value of the acceleration per unit filter bandwidth (1 Hz). The power spectral

density is expressed in units of $g^2/Hz$. The rms value of the acceleration within a frequency band between $f_1$ and $f_2$ is, with G being the power spectral density of acceleration:

$$A^2_{rms} = \int_{f_1}^{f_2} G(f)df \tag{54}$$

The value of the maximum acceleration at a given vibration frequency (actually a 1 Hz band of frequencies) is

$$A_{max}(f) = \sqrt{2} \cdot \sqrt{A^2_{rms}(f)}, \text{ (1 Hz)} \tag{55}$$

The ratio of the power in one of the 1 Hz wide sidebands to the carrier power, from the last section, is therefore (expressed in dB)

$$\mathcal{L}_v^{\,1}(f) = 10 \log (J_1(\sqrt{2}\; A(f)_{rms}\gamma F_o/f))/(J_o(\sqrt{2}\; A(f)_{rms}\gamma F_o/f))^2$$

$$\mathcal{L}_v^{\,1}(f) = 20 \log(\tfrac{1}{2}\sqrt{2}\; A_{rms}(f)\gamma F_o/f);\ 1 \text{ Hz} \tag{56}$$

Example 1: Assume

$$\begin{aligned} F_o &= 5.115 \text{ MHz} \\ \gamma &= 2 \times 10^{-9}/g. \end{aligned} \tag{57}$$

G(f) is as in Figure 6.

$$\mathcal{L}_v^{\,1}(f) = 20 \log(7.23 \times 10^{-3} A_{rms}(f)/f), \tag{58}$$

and is as shown in Figure 7.

## CONCLUSION

The preceding report describes the effects of vibration on crystal resonators. The $2 \times 10^{-9}$ per g used in the examples is the state-of-the-art for commercially available oscillators in 1979. Techniques in development such as the use of doubly rotated cuts[8,9], compound vibrators[10], and improved mounting and bonding techniques[11] promise to provide orders of magnitude improvement in vibration-resistance in the future.

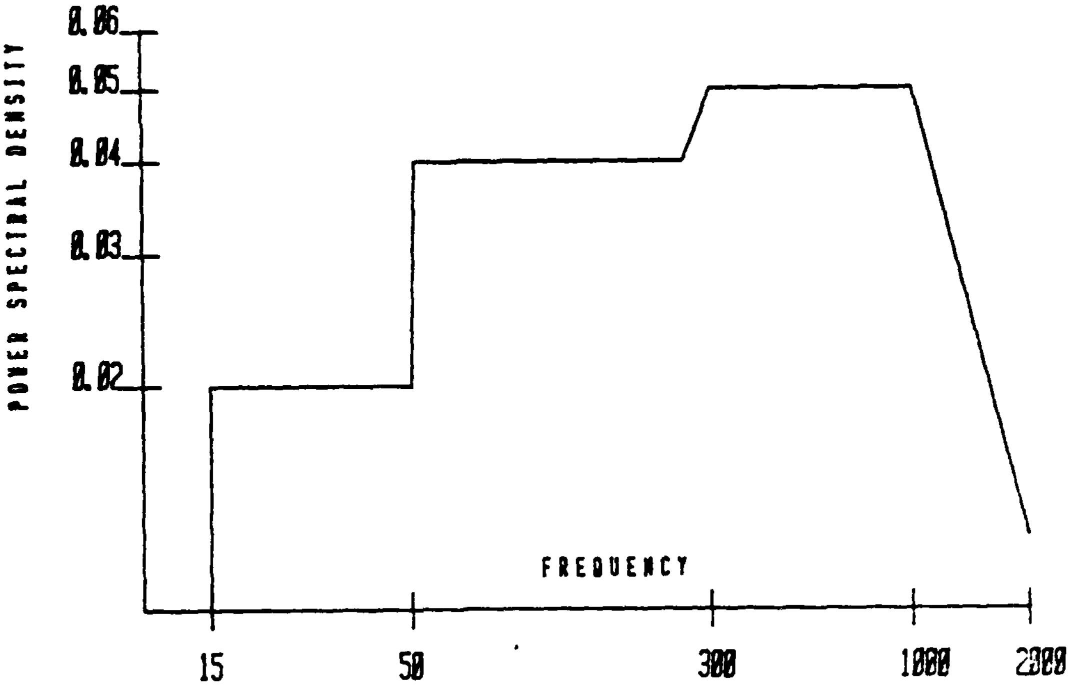


FIGURE 6 - AN EXAMPLE OF A RANDOM VIBRATION ENVELOPE


FIGURE 7 - NOISE EXPECTED FROM RANDOM VIBRATION
GIVEN IN FIGURE 6

## APPENDIX I

### FOURIER DECOMPOSITION OF $\cos(\beta \sin x)$ and $\sin(\beta \sin x)$

Problem: Fourier decompose $\cos(\beta \sin x)$ and $\sin(\beta \sin x)$. Consider the Maclaurin series expansion of $e^{\beta t/2}$ and $e^{-\beta/2t}$.

$$e^{\beta t/2} = \sum_{r=o}^{\infty} (1/r!)(\beta t/2)^r \tag{A1}$$

and

$$e^{-\beta/2t} = \sum_{s=o}^{\infty} (1/s!)(-\beta/2t)^s \quad t \neq 0 \tag{A2}$$

If we multiply (A1) by (A2) we have

$$(e^{\beta t/2})(e^{-\beta/2t}) = e^{(\beta/2)(t-t^{-1})} = \sum_{r=o}^{\infty} \sum_{s=o}^{\infty} ((-1)^s/(r!s!))(\beta/2)^{r+s} t^{r-s} \tag{A3}$$

The coefficient of the $t^o$ term is

$$1 - \beta^2/2^2 + \beta^4/(2^2 \cdot 4^2) - \beta^6/(2^2 \cdot 4^2 \cdot 6^2) + \ldots = J_o(\beta) \tag{A4}$$

The coefficient of the $t^{nth}$ term is

$$(\beta/2)^n/n! \{1 - (\beta/2)^2/(1 \cdot (n+1)) + (\beta/2)^4/(1 \cdot 2 \cdot (n+1)(n+2))\} \ldots = J_n(\beta) \tag{A5}$$

therefore

$$e^{\beta/2(t-t^{-1})} = J_o(\beta) + tJ_1(\beta) + t^2 J_2(\beta) + t^{-1} J_{-1}(\beta)$$

$$+ t^{-2} J_{-2}(\beta) + \ldots = \sum_{n=-\infty}^{\infty} t^n J_n(\beta) \tag{A6}$$

If we substitute $e^{ix}$ for t we get

$$e^{(\beta/2)(t-t^{-1})} = e^{(\beta/2)(e^{ix}-e^{-ix})} = e^{(i \beta \sin x)}$$

$$= \cos(\beta \sin x) + i \sin(\beta \sin x) \tag{A7}$$

so

$$\cos(\beta \sin x) + i \sin(\beta \sin x) = \sum_{n=-\infty}^{\infty} e^{inx} J_n(\beta)$$

$$= \sum_{n=-\infty}^{\infty} \{e^{inx} J_n(\beta) - e^{-inx} J_n(\beta)\} + J_o(\beta) \qquad (A8)$$

Using the identity

$$J_{-n}(\beta) = (-1)^n J_n(\beta) \qquad (A9)$$

we get

$$\cos(\beta \sin x) + i \sin(\beta \sin x) = J_o(\beta) + \sum_{n=1}^{\infty} (e^{inx} + (-1)^n e^{-inx}) J_n(\beta) \qquad (A10)$$

Using the relationship

$$\sin x = (e^{ix} - e^{-ix})/2i \qquad (A11)$$

and

$$\cos x = (e^{ix} + e^{-ix})/2 \qquad (A12)$$

we get from equation (A10)

$$\cos(\beta \sin x) + i \sin(\beta \sin x) = J_o(\beta) + 2 \sum_{k=1}^{\infty} J_{2k}(\beta) \cos(2kx)$$

$$+ 2i \sum_{k=o}^{\infty} J_{2k+1}(\beta) \sin((2k+1)x) \qquad (A13)$$

If we separate real and imaginary parts we have our result

$$\cos(\beta \sin x) = J_o(\beta) + 2 \sum_{k=1}^{\infty} J_{2k}(\beta) \cos(2kx) \qquad (A14)$$

and

$$\sin(\beta \sin x) = 2 \sum_{k=o}^{\infty} J_{2k+1}(\beta) \sin((2k+1)x) \qquad (A15)$$

## APPENDIX II

## BESSEL FUNCTIONS

Bessel Functions are defined as

$$J_p(\beta) = \sum_{k=o}^{\infty} ((-1)^k(\beta/2)^{2k+p}/k!\Gamma(p+k+1)) \tag{B1}$$

if p is an integer, the gamma function is then

$$\Gamma(p+k1) = (p+k)! \tag{B2}$$

therefore, for integer p

$$J_p(\beta) = \sum_{k=o}^{\infty} ((-1)^k(\beta/2)^{2k+p}/k!(p+k)!) \tag{B3}$$

Useful relations are

$$J_o(0) = 1 \text{ and } J_o(\beta<0.1) \simeq 1 \tag{B4}$$

and

$$J_{p>1}(0) = 0 \text{ and } J_1(\beta<0.1) \simeq \beta/2 \tag{B5}$$

Figure 8 is a plot of $J_o(\beta)$, $J_1(\beta)$ and $J_2(\beta)$ for $\beta = 0$ to 10. Figure 9 is a plot of $J_1(\beta)/J_o(\beta)$ and $J_2(\beta)/J_o(\beta)$ for $\beta = 0$ to 0.1. It can be seen that $J_1(\beta)/J_o(\beta)$ is $\simeq \beta/2$ and that $J_2(\beta)/J_o(\beta)$ is much less than $(\beta/2)$ in that region.

$J_o(\beta)$ has its first zero at 2.405. This is where $J_1(\beta)/J_o(\beta)$ goes to infinity.

Table I is a listing of values for $J_o$, $J_1$, $J_2$, $J_1/J_o$ and $J_2/J_o$ for $\beta = 0.001$ to 2.

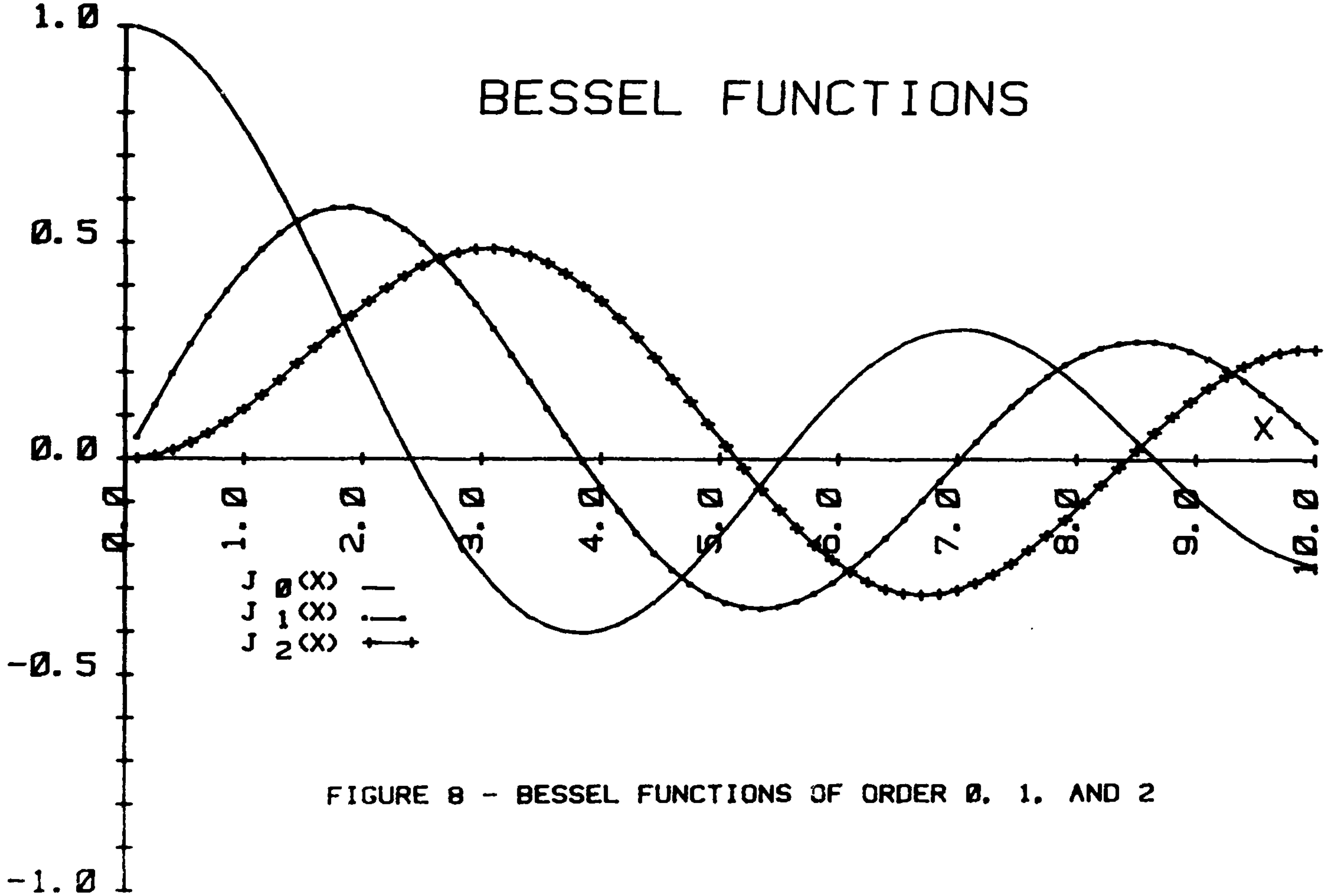


FIGURE 8 - BESSEL FUNCTIONS OF ORDER 0, 1, AND 2

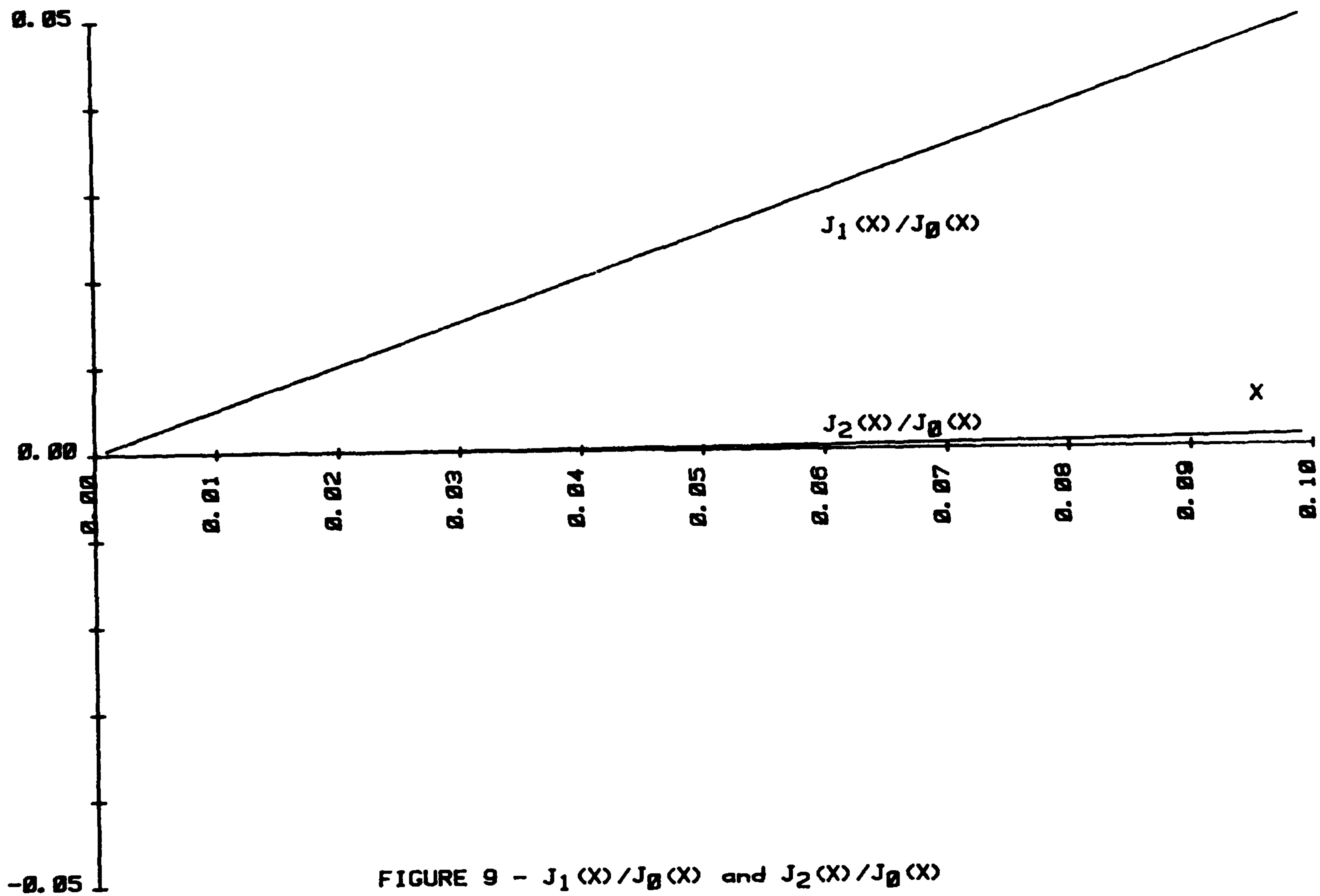


FIGURE 9 - $J_1(X)/J_0(X)$ and $J_2(X)/J_0(X)$

TABLE 1 - VALUES OF BESSEL FUNCTIONS

| X | J0(X) | J1(X) | J2(X) | J1/J0 | J2/J0 |
|---|---|---|---|---|---|
| 0.01 | 1.0000 | 0.0050 | 0.0000 | 5.000E-03 | 1.250E-05 |
| 0.02 | 0.9999 | 0.0100 | 0.0000 | 1.000E-02 | 5.000E-05 |
| 0.03 | 0.9998 | 0.0150 | 0.0001 | 1.500E-02 | 1.125E-04 |
| 0.04 | 0.9996 | 0.0200 | 0.0002 | 2.000E-02 | 2.001E-04 |
| 0.05 | 0.9994 | 0.0250 | 0.0003 | 2.501E-02 | 3.126E-04 |
| 0.06 | 0.9991 | 0.0300 | 0.0004 | 3.001E-02 | 4.503E-04 |
| 0.07 | 0.9988 | 0.0350 | 0.0006 | 3.502E-02 | 6.130E-04 |
| 0.08 | 0.9984 | 0.0400 | 0.0008 | 4.003E-02 | 8.009E-04 |
| 0.09 | 0.9980 | 0.0450 | 0.0010 | 4.505E-02 | 1.014E-03 |
| 0.10 | 0.9975 | 0.0499 | 0.0012 | 5.006E-02 | 1.252E-03 |
| 0.11 | 0.9970 | 0.0549 | 0.0015 | 5.508E-02 | 1.516E-03 |
| 0.12 | 0.9964 | 0.0599 | 0.0018 | 6.011E-02 | 1.804E-03 |
| 0.13 | 0.9958 | 0.0649 | 0.0021 | 6.514E-02 | 2.118E-03 |
| 0.14 | 0.9951 | 0.0698 | 0.0024 | 7.017E-02 | 2.458E-03 |
| 0.15 | 0.9944 | 0.0748 | 0.0028 | 7.521E-02 | 2.823E-03 |
| 0.16 | 0.9936 | 0.0797 | 0.0032 | 8.026E-02 | 3.214E-03 |
| 0.17 | 0.9928 | 0.0847 | 0.0036 | 8.531E-02 | 3.630E-03 |
| 0.18 | 0.9919 | 0.0896 | 0.0040 | 9.037E-02 | 4.072E-03 |
| 0.19 | 0.9910 | 0.0946 | 0.0045 | 9.543E-02 | 4.540E-03 |
| 0.20 | 0.9900 | 0.0995 | 0.0050 | 1.005E-01 | 5.034E-03 |
| 0.21 | 0.9890 | 0.1044 | 0.0055 | 1.056E-01 | 5.553E-03 |
| 0.22 | 0.9879 | 0.1093 | 0.0060 | 1.107E-01 | 6.099E-03 |
| 0.23 | 0.9868 | 0.1142 | 0.0066 | 1.158E-01 | 6.671E-03 |
| 0.24 | 0.9857 | 0.1191 | 0.0072 | 1.209E-01 | 7.270E-03 |
| 0.25 | 0.9844 | 0.1240 | 0.0078 | 1.260E-01 | 7.895E-03 |
| 0.26 | 0.9832 | 0.1289 | 0.0084 | 1.311E-01 | 8.546E-03 |
| 0.27 | 0.9819 | 0.1338 | 0.0091 | 1.362E-01 | 9.225E-03 |
| 0.28 | 0.9805 | 0.1386 | 0.0097 | 1.414E-01 | 9.930E-03 |
| 0.29 | 0.9791 | 0.1435 | 0.0104 | 1.465E-01 | 1.066E-02 |
| 0.30 | 0.9776 | 0.1483 | 0.0112 | 1.517E-01 | 1.142E-02 |
| 0.31 | 0.9761 | 0.1531 | 0.0119 | 1.569E-01 | 1.221E-02 |
| 0.32 | 0.9746 | 0.1580 | 0.0127 | 1.621E-01 | 1.302E-02 |
| 0.33 | 0.9730 | 0.1628 | 0.0135 | 1.673E-01 | 1.386E-02 |
| 0.34 | 0.9713 | 0.1676 | 0.0143 | 1.725E-01 | 1.473E-02 |
| 0.35 | 0.9696 | 0.1723 | 0.0152 | 1.777E-01 | 1.563E-02 |
| 0.36 | 0.9679 | 0.1771 | 0.0160 | 1.830E-01 | 1.656E-02 |
| 0.37 | 0.9661 | 0.1819 | 0.0169 | 1.882E-01 | 1.751E-02 |
| 0.38 | 0.9642 | 0.1866 | 0.0178 | 1.935E-01 | 1.850E-02 |
| 0.39 | 0.9623 | 0.1913 | 0.0188 | 1.988E-01 | 1.951E-02 |
| 0.40 | 0.9604 | 0.1960 | 0.0197 | 2.041E-01 | 2.055E-02 |
| 0.41 | 0.9584 | 0.2007 | 0.0207 | 2.094E-01 | 2.162E-02 |
| 0.42 | 0.9564 | 0.2054 | 0.0217 | 2.148E-01 | 2.272E-02 |
| 0.43 | 0.9543 | 0.2101 | 0.0228 | 2.201E-01 | 2.385E-02 |
| 0.44 | 0.9522 | 0.2147 | 0.0238 | 2.255E-01 | 2.501E-02 |
| 0.45 | 0.9500 | 0.2194 | 0.0249 | 2.309E-01 | 2.620E-02 |
| 0.46 | 0.9478 | 0.2240 | 0.0260 | 2.363E-01 | 2.742E-02 |
| 0.47 | 0.9455 | 0.2286 | 0.0271 | 2.417E-01 | 2.867E-02 |
| 0.48 | 0.9432 | 0.2332 | 0.0283 | 2.472E-01 | 2.995E-02 |
| 0.49 | 0.9409 | 0.2377 | 0.0294 | 2.527E-01 | 3.127E-02 |
| 0.50 | 0.9385 | 0.2423 | 0.0306 | 2.582E-01 | 3.261E-02 |

TABLE 1 (CONTINUED)

| X | J0(X) | J1(X) | J2(X) | J1/J0 | J2/J0 |
|---|---|---|---|---|---|
| 0.51 | 0.9360 | 0.2468 | 0.0318 | 2.637E-01 | 3.399E-02 |
| 0.52 | 0.9335 | 0.2513 | 0.0330 | 2.692E-01 | 3.540E-02 |
| 0.53 | 0.9310 | 0.2558 | 0.0343 | 2.748E-01 | 3.684E-02 |
| 0.54 | 0.9284 | 0.2603 | 0.0356 | 2.803E-01 | 3.831E-02 |
| 0.55 | 0.9258 | 0.2647 | 0.0369 | 2.860E-01 | 3.982E-02 |
| 0.56 | 0.9231 | 0.2692 | 0.0382 | 2.916E-01 | 4.137E-02 |
| 0.57 | 0.9204 | 0.2736 | 0.0395 | 2.972E-01 | 4.294E-02 |
| 0.58 | 0.9177 | 0.2780 | 0.0409 | 3.029E-01 | 4.455E-02 |
| 0.59 | 0.9149 | 0.2823 | 0.0423 | 3.086E-01 | 4.620E-02 |
| 0.60 | 0.9120 | 0.2867 | 0.0437 | 3.144E-01 | 4.788E-02 |
| 0.61 | 0.9091 | 0.2910 | 0.0451 | 3.201E-01 | 4.959E-02 |
| 0.62 | 0.9062 | 0.2953 | 0.0465 | 3.259E-01 | 5.135E-02 |
| 0.63 | 0.9032 | 0.2996 | 0.0480 | 3.317E-01 | 5.313E-02 |
| 0.64 | 0.9002 | 0.3039 | 0.0495 | 3.376E-01 | 5.496E-02 |
| 0.65 | 0.8971 | 0.3081 | 0.0510 | 3.435E-01 | 5.682E-02 |
| 0.66 | 0.8940 | 0.3124 | 0.0525 | 3.494E-01 | 5.872E-02 |
| 0.67 | 0.8909 | 0.3166 | 0.0540 | 3.553E-01 | 6.066E-02 |
| 0.68 | 0.8877 | 0.3207 | 0.0556 | 3.613E-01 | 6.264E-02 |
| 0.69 | 0.8845 | 0.3249 | 0.0572 | 3.673E-01 | 6.466E-02 |
| 0.70 | 0.8812 | 0.3290 | 0.0588 | 3.733E-01 | 6.671E-02 |
| 0.71 | 0.8779 | 0.3331 | 0.0604 | 3.794E-01 | 6.881E-02 |
| 0.72 | 0.8745 | 0.3372 | 0.0620 | 3.855E-01 | 7.095E-02 |
| 0.73 | 0.8711 | 0.3412 | 0.0637 | 3.917E-01 | 7.313E-02 |
| 0.74 | 0.8677 | 0.3452 | 0.0654 | 3.979E-01 | 7.535E-02 |
| 0.75 | 0.8642 | 0.3492 | 0.0671 | 4.041E-01 | 7.761E-02 |
| 0.76 | 0.8607 | 0.3532 | 0.0688 | 4.104E-01 | 7.992E-02 |
| 0.77 | 0.8572 | 0.3572 | 0.0705 | 4.167E-01 | 8.227E-02 |
| 0.78 | 0.8536 | 0.3611 | 0.0723 | 4.230E-01 | 8.466E-02 |
| 0.79 | 0.8500 | 0.3650 | 0.0740 | 4.294E-01 | 8.710E-02 |
| 0.80 | 0.8463 | 0.3688 | 0.0758 | 4.358E-01 | 8.959E-02 |
| 0.81 | 0.8426 | 0.3727 | 0.0776 | 4.423E-01 | 9.212E-02 |
| 0.82 | 0.8388 | 0.3765 | 0.0794 | 4.488E-01 | 9.470E-02 |
| 0.83 | 0.8350 | 0.3803 | 0.0813 | 4.554E-01 | 9.733E-02 |
| 0.84 | 0.8312 | 0.3840 | 0.0831 | 4.620E-01 | 1.000E-01 |
| 0.85 | 0.8274 | 0.3878 | 0.0850 | 4.687E-01 | 1.027E-01 |
| 0.86 | 0.8235 | 0.3915 | 0.0869 | 4.754E-01 | 1.055E-01 |
| 0.87 | 0.8195 | 0.3951 | 0.0888 | 4.821E-01 | 1.083E-01 |
| 0.88 | 0.8156 | 0.3988 | 0.0907 | 4.889E-01 | 1.112E-01 |
| 0.89 | 0.8116 | 0.4024 | 0.0926 | 4.958E-01 | 1.141E-01 |
| 0.90 | 0.8075 | 0.4059 | 0.0946 | 5.027E-01 | 1.171E-01 |
| 0.91 | 0.8034 | 0.4095 | 0.0966 | 5.097E-01 | 1.202E-01 |
| 0.92 | 0.7993 | 0.4130 | 0.0985 | 5.167E-01 | 1.233E-01 |
| 0.93 | 0.7952 | 0.4165 | 0.1005 | 5.238E-01 | 1.264E-01 |
| 0.94 | 0.7910 | 0.4200 | 0.1025 | 5.309E-01 | 1.296E-01 |
| 0.95 | 0.7868 | 0.4234 | 0.1046 | 5.381E-01 | 1.329E-01 |
| 0.96 | 0.7825 | 0.4268 | 0.1066 | 5.454E-01 | 1.362E-01 |
| 0.97 | 0.7783 | 0.4302 | 0.1087 | 5.527E-01 | 1.396E-01 |
| 0.98 | 0.7739 | 0.4335 | 0.1107 | 5.601E-01 | 1.431E-01 |
| 0.99 | 0.7696 | 0.4368 | 0.1128 | 5.676E-01 | 1.466E-01 |
| 1.00 | 0.7652 | 0.4401 | 0.1149 | 5.751E-01 | 1.502E-01 |

TABLE 1 (CONTINUED)

| X | J0(X) | J1(X) | J2(X) | J1/J0 | J2/J0 |
|---|---|---|---|---|---|
| 1.01 | 0.7608 | 0.4433 | 0.1170 | 5.827E-01 | 1.538E-01 |
| 1.02 | 0.7563 | 0.4465 | 0.1191 | 5.903E-01 | 1.575E-01 |
| 1.03 | 0.7519 | 0.4497 | 0.1213 | 5.981E-01 | 1.613E-01 |
| 1.04 | 0.7473 | 0.4528 | 0.1234 | 6.059E-01 | 1.651E-01 |
| 1.05 | 0.7428 | 0.4559 | 0.1256 | 6.138E-01 | 1.691E-01 |
| 1.06 | 0.7382 | 0.4590 | 0.1278 | 6.217E-01 | 1.731E-01 |
| 1.07 | 0.7336 | 0.4620 | 0.1299 | 6.298E-01 | 1.771E-01 |
| 1.08 | 0.7290 | 0.4650 | 0.1321 | 6.379E-01 | 1.813E-01 |
| 1.09 | 0.7243 | 0.4680 | 0.1343 | 6.461E-01 | 1.855E-01 |
| 1.10 | 0.7196 | 0.4709 | 0.1366 | 6.544E-01 | 1.898E-01 |
| 1.11 | 0.7149 | 0.4738 | 0.1388 | 6.628E-01 | 1.941E-01 |
| 1.12 | 0.7101 | 0.4767 | 0.1410 | 6.712E-01 | 1.986E-01 |
| 1.13 | 0.7054 | 0.4795 | 0.1433 | 6.798E-01 | 2.031E-01 |
| 1.14 | 0.7006 | 0.4823 | 0.1456 | 6.884E-01 | 2.078E-01 |
| 1.15 | 0.6957 | 0.4850 | 0.1478 | 6.972E-01 | 2.125E-01 |
| 1.16 | 0.6909 | 0.4878 | 0.1501 | 7.060E-01 | 2.173E-01 |
| 1.17 | 0.6860 | 0.4904 | 0.1524 | 7.150E-01 | 2.222E-01 |
| 1.18 | 0.6810 | 0.4931 | 0.1547 | 7.240E-01 | 2.272E-01 |
| 1.19 | 0.6761 | 0.4957 | 0.1570 | 7.332E-01 | 2.323E-01 |
| 1.20 | 0.6711 | 0.4983 | 0.1593 | 7.425E-01 | 2.374E-01 |
| 1.21 | 0.6661 | 0.5008 | 0.1617 | 7.518E-01 | 2.427E-01 |
| 1.22 | 0.6611 | 0.5033 | 0.1640 | 7.613E-01 | 2.481E-01 |
| 1.23 | 0.6561 | 0.5058 | 0.1664 | 7.710E-01 | 2.536E-01 |
| 1.24 | 0.6510 | 0.5082 | 0.1687 | 7.807E-01 | 2.592E-01 |
| 1.25 | 0.6459 | 0.5106 | 0.1711 | 7.906E-01 | 2.649E-01 |
| 1.26 | 0.6408 | 0.5130 | 0.1735 | 8.005E-01 | 2.707E-01 |
| 1.27 | 0.6356 | 0.5153 | 0.1758 | 8.107E-01 | 2.766E-01 |
| 1.28 | 0.6305 | 0.5176 | 0.1782 | 8.209E-01 | 2.827E-01 |
| 1.29 | 0.6253 | 0.5198 | 0.1806 | 8.313E-01 | 2.889E-01 |
| 1.30 | 0.6201 | 0.5220 | 0.1830 | 8.419E-01 | 2.952E-01 |
| 1.31 | 0.6149 | 0.5242 | 0.1854 | 8.525E-01 | 3.016E-01 |
| 1.32 | 0.6096 | 0.5263 | 0.1878 | 8.634E-01 | 3.081E-01 |
| 1.33 | 0.6043 | 0.5284 | 0.1903 | 8.744E-01 | 3.148E-01 |
| 1.34 | 0.5990 | 0.5305 | 0.1927 | 8.855E-01 | 3.217E-01 |
| 1.35 | 0.5937 | 0.5325 | 0.1951 | 8.968E-01 | 3.286E-01 |
| 1.36 | 0.5884 | 0.5344 | 0.1976 | 9.083E-01 | 3.358E-01 |
| 1.37 | 0.5830 | 0.5364 | 0.2000 | 9.200E-01 | 3.430E-01 |
| 1.38 | 0.5777 | 0.5383 | 0.2024 | 9.318E-01 | 3.505E-01 |
| 1.39 | 0.5723 | 0.5401 | 0.2049 | 9.438E-01 | 3.581E-01 |
| 1.40 | 0.5669 | 0.5419 | 0.2074 | 9.561E-01 | 3.658E-01 |
| 1.41 | 0.5614 | 0.5437 | 0.2098 | 9.685E-01 | 3.737E-01 |
| 1.42 | 0.5560 | 0.5455 | 0.2123 | 9.811E-01 | 3.818E-01 |
| 1.43 | 0.5505 | 0.5472 | 0.2147 | 9.939E-01 | 3.901E-01 |
| 1.44 | 0.5450 | 0.5488 | 0.2172 | 1.007E 00 | 3.985E-01 |
| 1.45 | 0.5395 | 0.5504 | 0.2197 | 1.020E 00 | 4.072E-01 |
| 1.46 | 0.5340 | 0.5520 | 0.2222 | 1.034E 00 | 4.160E-01 |
| 1.47 | 0.5285 | 0.5536 | 0.2246 | 1.047E 00 | 4.251E-01 |
| 1.48 | 0.5230 | 0.5551 | 0.2271 | 1.061E 00 | 4.343E-01 |
| 1.49 | 0.5174 | 0.5565 | 0.2296 | 1.076E 00 | 4.438E-01 |
| 1.50 | 0.5118 | 0.5579 | 0.2321 | 1.090E 00 | 4.534E-01 |

TABLE 1 (CONTINUED)

| X | J0(X) | J1(X) | J2(X) | J1/J0 | J2/J0 |
|---|---|---|---|---|---|
| 1.51 | 0.5062 | 0.5593 | 0.2346 | 1.105E 00 | 4.634E-01 |
| 1.52 | 0.5006 | 0.5607 | 0.2371 | 1.120E 00 | 4.735E-01 |
| 1.53 | 0.4950 | 0.5620 | 0.2395 | 1.135E 00 | 4.839E-01 |
| 1.54 | 0.4894 | 0.5632 | 0.2420 | 1.151E 00 | 4.946E-01 |
| 1.55 | 0.4838 | 0.5644 | 0.2445 | 1.167E 00 | 5.055E-01 |
| 1.56 | 0.4781 | 0.5656 | 0.2470 | 1.183E 00 | 5.166E-01 |
| 1.57 | 0.4725 | 0.5667 | 0.2495 | 1.200E 00 | 5.281E-01 |
| 1.58 | 0.4668 | 0.5678 | 0.2520 | 1.216E 00 | 5.399E-01 |
| 1.59 | 0.4611 | 0.5689 | 0.2545 | 1.234E 00 | 5.519E-01 |
| 1.60 | 0.4554 | 0.5699 | 0.2570 | 1.251E 00 | 5.643E-01 |
| 1.61 | 0.4497 | 0.5709 | 0.2595 | 1.269E 00 | 5.770E-01 |
| 1.62 | 0.4440 | 0.5718 | 0.2619 | 1.288E 00 | 5.900E-01 |
| 1.63 | 0.4383 | 0.5727 | 0.2644 | 1.307E 00 | 6.033E-01 |
| 1.64 | 0.4325 | 0.5735 | 0.2669 | 1.326E 00 | 6.171E-01 |
| 1.65 | 0.4268 | 0.5743 | 0.2694 | 1.346E 00 | 6.312E-01 |
| 1.66 | 0.4210 | 0.5751 | 0.2719 | 1.366E 00 | 6.457E-01 |
| 1.67 | 0.4153 | 0.5758 | 0.2743 | 1.387E 00 | 6.606E-01 |
| 1.68 | 0.4095 | 0.5765 | 0.2768 | 1.408E 00 | 6.759E-01 |
| 1.69 | 0.4038 | 0.5772 | 0.2793 | 1.429E 00 | 6.917E-01 |
| 1.70 | 0.3980 | 0.5778 | 0.2817 | 1.452E 00 | 7.079E-01 |
| 1.71 | 0.3922 | 0.5783 | 0.2842 | 1.475E 00 | 7.246E-01 |
| 1.72 | 0.3864 | 0.5788 | 0.2867 | 1.498E 00 | 7.418E-01 |
| 1.73 | 0.3806 | 0.5793 | 0.2891 | 1.522E 00 | 7.596E-01 |
| 1.74 | 0.3748 | 0.5798 | 0.2916 | 1.547E 00 | 7.778E-01 |
| 1.75 | 0.3690 | 0.5802 | 0.2940 | 1.572E 00 | 7.967E-01 |
| 1.76 | 0.3632 | 0.5805 | 0.2964 | 1.598E 00 | 8.161E-01 |
| 1.77 | 0.3574 | 0.5808 | 0.2989 | 1.625E 00 | 8.362E-01 |
| 1.78 | 0.3516 | 0.5811 | 0.3013 | 1.653E 00 | 8.569E-01 |
| 1.79 | 0.3458 | 0.5813 | 0.3037 | 1.681E 00 | 8.783E-01 |
| 1.80 | 0.3400 | 0.5815 | 0.3061 | 1.710E 00 | 9.005E-01 |
| 1.81 | 0.3342 | 0.5817 | 0.3086 | 1.741E 00 | 9.233E-01 |
| 1.82 | 0.3284 | 0.5818 | 0.3110 | 1.772E 00 | 9.470E-01 |
| 1.83 | 0.3225 | 0.5818 | 0.3134 | 1.804E 00 | 9.715E-01 |
| 1.84 | 0.3167 | 0.5819 | 0.3157 | 1.837E 00 | 9.969E-01 |
| 1.85 | 0.3109 | 0.5818 | 0.3181 | 1.872E 00 | 1.023E 00 |
| 1.86 | 0.3051 | 0.5818 | 0.3205 | 1.907E 00 | 1.051E 00 |
| 1.87 | 0.2993 | 0.5817 | 0.3229 | 1.944E 00 | 1.079E 00 |
| 1.88 | 0.2934 | 0.5816 | 0.3252 | 1.982E 00 | 1.108E 00 |
| 1.89 | 0.2876 | 0.5814 | 0.3276 | 2.021E 00 | 1.139E 00 |
| 1.90 | 0.2818 | 0.5812 | 0.3299 | 2.062E 00 | 1.171E 00 |
| 1.91 | 0.2760 | 0.5809 | 0.3323 | 2.105E 00 | 1.204E 00 |
| 1.92 | 0.2702 | 0.5806 | 0.3346 | 2.149E 00 | 1.238E 00 |
| 1.93 | 0.2644 | 0.5803 | 0.3369 | 2.195E 00 | 1.274E 00 |
| 1.94 | 0.2586 | 0.5799 | 0.3392 | 2.242E 00 | 1.312E 00 |
| 1.95 | 0.2528 | 0.5794 | 0.3415 | 2.292E 00 | 1.351E 00 |
| 1.96 | 0.2470 | 0.5790 | 0.3438 | 2.344E 00 | 1.392E 00 |
| 1.97 | 0.2412 | 0.5785 | 0.3461 | 2.398E 00 | 1.435E 00 |
| 1.98 | 0.2354 | 0.5779 | 0.3483 | 2.455E 00 | 1.480E 00 |
| 1.99 | 0.2297 | 0.5773 | 0.3506 | 2.514E 00 | 1.527E 00 |
| 2.00 | 0.2239 | 0.5767 | 0.3528 | 2.576E 00 | 1.576E 00 |

TABLE 1 (CONTINUED)

| X | J0(X) | J1(X) | J2(X) | J1/J0 | J2/J0 |
|---|---|---|---|---|---|
| 2.01 | 0.2181 | 0.5761 | 0.3551 | 2.641E 00 | 1.628E 00 |
| 2.02 | 0.2124 | 0.5754 | 0.3573 | 2.709E 00 | 1.682E 00 |
| 2.03 | 0.2066 | 0.5746 | 0.3595 | 2.781E 00 | 1.740E 00 |
| 2.04 | 0.2009 | 0.5738 | 0.3617 | 2.857E 00 | 1.801E 00 |
| 2.05 | 0.1951 | 0.5730 | 0.3639 | 2.936E 00 | 1.865E 00 |
| 2.06 | 0.1894 | 0.5721 | 0.3661 | 3.021E 00 | 1.933E 00 |
| 2.07 | 0.1837 | 0.5712 | 0.3682 | 3.110E 00 | 2.004E 00 |
| 2.08 | 0.1780 | 0.5703 | 0.3704 | 3.204E 00 | 2.081E 00 |
| 2.09 | 0.1723 | 0.5693 | 0.3725 | 3.304E 00 | 2.162E 00 |
| 2.10 | 0.1666 | 0.5683 | 0.3746 | 3.411E 00 | 2.249E 00 |
| 2.11 | 0.1609 | 0.5672 | 0.3767 | 3.525E 00 | 2.341E 00 |
| 2.12 | 0.1553 | 0.5661 | 0.3788 | 3.646E 00 | 2.440E 00 |
| 2.13 | 0.1496 | 0.5650 | 0.3809 | 3.777E 00 | 2.546E 00 |
| 2.14 | 0.1440 | 0.5638 | 0.3830 | 3.916E 00 | 2.660E 00 |
| 2.15 | 0.1383 | 0.5626 | 0.3850 | 4.067E 00 | 2.783E 00 |
| 2.16 | 0.1327 | 0.5614 | 0.3871 | 4.230E 00 | 2.917E 00 |
| 2.17 | 0.1271 | 0.5601 | 0.3891 | 4.406E 00 | 3.062E 00 |
| 2.18 | 0.1215 | 0.5587 | 0.3911 | 4.598E 00 | 3.219E 00 |
| 2.19 | 0.1159 | 0.5574 | 0.3931 | 4.808E 00 | 3.391E 00 |
| 2.20 | 0.1104 | 0.5560 | 0.3951 | 5.038E 00 | 3.58[illegible]E 00 |
| 2.21 | 0.1048 | 0.5545 | 0.3970 | 5.291E 00 | 3.78[illegible]E 00 |
| 2.22 | 0.0993 | 0.5530 | 0.3990 | 5.571E 00 | 4.019E 00 |
| 2.23 | 0.0937 | 0.5515 | 0.4009 | 5.883E 00 | 4.276E 00 |
| 2.24 | 0.0882 | 0.5500 | 0.4028 | 6.233E 00 | 4.565E 00 |
| 2.25 | 0.0827 | 0.5484 | 0.4047 | 6.627E 00 | 4.891E 00 |
| 2.26 | 0.0773 | 0.5468 | 0.4066 | 7.075E 00 | 5.261E 00 |
| 2.27 | 0.0718 | 0.5451 | 0.4084 | 7.590E 00 | 5.687E 00 |
| 2.28 | 0.0664 | 0.5434 | 0.4103 | 8.187E 00 | 6.181E 00 |
| 2.29 | 0.0609 | 0.5416 | 0.4121 | 8.887E 00 | 6.762E 00 |
| 2.30 | 0.0555 | 0.5399 | 0.4139 | 9.720E 00 | 7.4[illegible]3E 00 |
| 2.31 | 0.0502 | 0.5381 | 0.4157 | 1.073E 01 | 8.269E 00 |
| 2.32 | 0.0448 | 0.5362 | 0.4175 | 1.197E 01 | 9.323E 00 |
| 2.33 | 0.0394 | 0.5343 | 0.4192 | 1.355E 01 | 1.06[illegible]E 01 |
| 2.34 | 0.0341 | 0.5324 | 0.4210 | 1.562E 01 | 1.2[illegible]E 01 |
| 2.35 | 0.0288 | 0.5305 | 0.4227 | 1.843E 01 | [illegible].4[illegible]E 01 |
| 2.36 | 0.0235 | 0.5285 | 0.4244 | 2.250E 01 | 1.6[illegible]7E 01 |
| 2.37 | 0.0182 | 0.5265 | 0.4261 | 2.891E 01 | 2.34[illegible]E 01 |
| 2.38 | 0.0130 | 0.5244 | 0.4277 | 4.048E 01 | 3.[illegible]E 01 |
| 2.39 | 0.0077 | 0.5223 | 0.4294 | 6.766E 01 | [illegible].5[illegible]E 01 |
| 2.40 | 0.0025 | 0.5202 | 0.4310 | 2.074E 02 | 1.719E 0[illegible] |
| 2.41 | -0.0027 | 0.5180 | 0.4326 | -1.930E 02 | -1.6[illegible]E 02 |
| 2.42 | -0.0079 | 0.5158 | 0.4342 | -6.569E 01 | -[illegible] |
| 2.43 | -0.0130 | 0.5136 | 0.4357 | -3.951E 01 | -[illegible].3[illegible]E 01 |
| 2.44 | -0.0181 | 0.5113 | 0.4373 | -2.821E 01 | -2.41[illegible] |
| 2.45 | -0.0232 | 0.5091 | 0.4388 | -2.192E 01 | -1.8[illegible]E 01 |
| 2.46 | -0.0283 | 0.5067 | 0.4403 | -1.790E 01 | -1.55[illegible] |
| 2.47 | -0.0334 | 0.5044 | 0.4418 | -1.512E 01 | -1.324E 01 |
| 2.48 | -0.0384 | 0.5020 | 0.4432 | -1.307E 01 | -1.1[illegible]4E [illegible] |
| 2.49 | -0.0434 | 0.4996 | 0.4446 | -1.151E 01 | -1.02[illegible] |
| 2.50 | -0.0484 | 0.4971 | 0.4461 | -1.027E 01 | -9.[illegible]19E 00 |

TABLE 1 (CONTINUED)

| X | J0(X) | J1(X) | J2(X) | J1/J0 | J2/J0 |
|---|---|---|---|---|---|
| 2.51 | -0.0533 | 0.4946 | 0.4475 | -9.272E 00 | -8.388E 00 |
| 2.52 | -0.0583 | 0.4921 | 0.4488 | -8.444E 00 | -7.702E 00 |
| 2.53 | -0.0632 | 0.4895 | 0.4502 | -7.748E 00 | -7.125E 00 |
| 2.54 | -0.0681 | 0.4870 | 0.4515 | -7.154E 00 | -6.633E 00 |
| 2.55 | -0.0729 | 0.4843 | 0.4528 | -6.642E 00 | -6.209E 00 |
| 2.56 | -0.0778 | 0.4817 | 0.4541 | -6.195E 00 | -5.840E 00 |
| 2.57 | -0.0826 | 0.4790 | 0.4553 | -5.802E 00 | -5.515E 00 |
| 2.58 | -0.0873 | 0.4763 | 0.4566 | -5.454E 00 | -5.228E 00 |
| 2.59 | -0.0921 | 0.4736 | 0.4578 | -5.143E 00 | -4.971E 00 |
| 2.60 | -0.0968 | 0.4708 | 0.4590 | -4.864E 00 | -4.741E 00 |
| 2.61 | -0.1015 | 0.4680 | 0.4601 | -4.611E 00 | -4.533E 00 |
| 2.62 | -0.1062 | 0.4652 | 0.4613 | -4.382E 00 | -4.345E 00 |
| 2.63 | -0.1108 | 0.4624 | 0.4624 | -4.173E 00 | -4.173E 00 |
| 2.64 | -0.1154 | 0.4595 | 0.4635 | -3.981E 00 | -4.016E 00 |
| 2.65 | -0.1200 | 0.4566 | 0.4646 | -3.805E 00 | -3.872E 00 |
| 2.66 | -0.1245 | 0.4536 | 0.4656 | -3.642E 00 | -3.739E 00 |
| 2.67 | -0.1291 | 0.4507 | 0.4666 | -3.492E 00 | -3.616E 00 |
| 2.68 | -0.1336 | 0.4477 | 0.4676 | -3.352E 00 | -3.501E 00 |
| 2.69 | -0.1380 | 0.4446 | 0.4686 | -3.222E 00 | -3.395E 00 |
| 2.70 | -0.1424 | 0.4416 | 0.4696 | -3.100E 00 | -3.296E 00 |
| 2.71 | -0.1469 | 0.4385 | 0.4705 | -2.986E 00 | -3.204E 00 |
| 2.72 | -0.1512 | 0.4354 | 0.4714 | -2.879E 00 | -3.117E 00 |
| 2.73 | -0.1556 | 0.4323 | 0.4723 | -2.779E 00 | -3.036E 00 |
| 2.74 | -0.1599 | 0.4291 | 0.4731 | -2.684E 00 | -2.959E 00 |
| 2.75 | -0.1641 | 0.4260 | 0.4739 | -2.595E 00 | -2.887E 00 |
| 2.76 | -0.1684 | 0.4228 | 0.4747 | -2.511E 00 | -2.819E 00 |
| 2.77 | -0.1726 | 0.4195 | 0.4755 | -2.431E 00 | -2.755E 00 |
| 2.78 | -0.1768 | 0.4163 | 0.4763 | -2.355E 00 | -2.694E 00 |
| 2.79 | -0.1809 | 0.4130 | 0.4770 | -2.283E 00 | -2.636E 00 |
| 2.80 | -0.1850 | 0.4097 | 0.4777 | -2.214E 00 | -2.582E 00 |
| 2.81 | -0.1891 | 0.4064 | 0.4784 | -2.149E 00 | -2.529E 00 |
| 2.82 | -0.1932 | 0.4030 | 0.4790 | -2.086E 00 | -2.480E 00 |
| 2.83 | -0.1972 | 0.3997 | 0.4796 | -2.027E 00 | -2.432E 00 |
| 2.84 | -0.2012 | 0.3963 | 0.4802 | -1.970E 00 | -2.387E 00 |
| 2.85 | -0.2051 | 0.3928 | 0.4808 | -1.915E 00 | -2.344E 00 |
| 2.86 | -0.2090 | 0.3894 | 0.4813 | -1.863E 00 | -2.303E 00 |
| 2.87 | -0.2129 | 0.3859 | 0.4818 | -1.813E 00 | -2.263E 00 |
| 2.88 | -0.2167 | 0.3825 | 0.4823 | -1.765E 00 | -2.225E 00 |
| 2.89 | -0.2205 | 0.3790 | 0.4828 | -1.718E 00 | -2.189E 00 |
| 2.90 | -0.2243 | 0.3754 | 0.4832 | -1.674E 00 | -2.154E 00 |
| 2.91 | -0.2280 | 0.3719 | 0.4836 | -1.631E 00 | -2.121E 00 |
| 2.92 | -0.2317 | 0.3683 | 0.4840 | -1.589E 00 | -2.089E 00 |
| 2.93 | -0.2354 | 0.3647 | 0.4844 | -1.549E 00 | -2.058E 00 |
| 2.94 | -0.2390 | 0.3611 | 0.4847 | -1.511E 00 | -2.028E 00 |
| 2.95 | -0.2426 | 0.3575 | 0.4850 | -1.473E 00 | -1.999E 00 |
| 2.96 | -0.2462 | 0.3538 | 0.4853 | -1.437E 00 | -1.971E 00 |
| 2.97 | -0.2497 | 0.3502 | 0.4855 | -1.402E 00 | -1.944E 00 |
| 2.98 | -0.2532 | 0.3465 | 0.4857 | -1.368E 00 | -1.918E 00 |
| 2.99 | -0.2566 | 0.3428 | 0.4859 | -1.336E 00 | -1.893E 00 |
| 3.00 | -0.2601 | 0.3391 | 0.4861 | -1.304E 00 | -1.869E 00 |

APPENDIX III

COMPLEX VIBRATIONS

Calculations Leading to the Result for Complex Vibrations

We have from equation 52

$$V(t) = V_o\{\cos(2\pi F_o t)\ \cos((\Delta F_1/f_{v1})\ \sin(2\pi f_{v1} t))$$

$$+ (\Delta F_2/f_{v2})\ \sin(2\pi f_{v2} t)) -$$

$$\sin(2\pi F_o t)\ \sin((\Delta F_1/f_{v1})\ \sin(2\pi f_{v1} t))$$

$$+ (\Delta F_2/f_{v2})\ \sin(2\pi f_{v2} t)\} \tag{C1}$$

Using equation (12) on (C1) and substituting

$$X_1 = (\Delta F_1/f_{v1})\ \sin(2\pi f t)t) \tag{C2}$$

$$X_2 = (\Delta F_2/f_{v2})\ \sin(2\pi f_{v2} t) \tag{C3}$$

we get

$$V(t) = V_o\{\cos(2\pi F_o t)\ \cos X_1\ \cos X_2 - \cos(2\pi F_o t)\ \sin X_1\ \sin X_2$$

$$- \sin(2\pi F_o t)\ \sin X_1\ \cos X_2 - \sin(2\pi F_o t)\ \cos X_1\ \sin X_2\ . \tag{C4}$$

If $\Delta F_1/f_{v1}$ and $\Delta F_2/f_{v2} \ll 1$ and substituting $\beta_1$ for $\Delta F_1/f_{v1}$ and $\beta_2$ for $\Delta F_2/f_{v2}$ we get

$$\cos(X_i) = J_o(\beta_i)\quad (i = 1,2) \tag{C5}$$

and

$$\sin(X_i) = 2J_1(\beta_i)\ \sin(2\pi f_{vi} t)(i = 1,2)\ . \tag{C6}$$

Using (C5) and (C6) in (C4) we get

$$V(t) = V_o\{J_o(\beta_1)\ J_o(\beta_2)\ \cos(2\pi F_o t)$$

$$- 4\ J_1(\beta_1)\ J_1(\beta_2)\ \sin(2\pi f_{v1} t)\ \sin(2\pi f_{v2} t)\ \cos(2\pi F_o t)$$

$$- 2\ J_1(\beta_1)\ J_1(\beta_2)\ \sin(2\pi f_{v1} t)\ \sin(2\pi F_o t)$$

$$+ 2\ J_o(\beta_1)\ J_1(\beta_2)\ \sin(2\pi f_{v2} t)\ \sin(2\pi F_o t)\} \qquad (C7)$$

The second term can be decomposed by

$$\cos A \sin B \sin C = \cos A(-\tfrac{1}{2}\cos(B+C) + \tfrac{1}{2}(\cos(B-C))$$

$$= \tfrac{1}{2}\cos A \cos(B-C) - \tfrac{1}{2}\cos A(\cos(B+C))$$

$$= \tfrac{1}{4}\{\cos(A+B+C) + \cos(A+B-C)\,.$$

$$- \cos(A-B-C) - \cos(A+B+C)\} \qquad (C8)$$

So, using (C8) in (C7), we get

$$V(t) = V_o\ J_o(\beta_1)\ J_o(\beta_2)\ \cos(2\pi F_o t)$$

$$+ J_1(\beta_1)\ J_o(\beta_2)(\cos(2\pi(F_o+f_{v1})t - \cos(2\pi F_o - f_{v1})t)$$

$$+ J_o(\beta_1)\ J_1(\beta_2)(\cos(2\pi(F_o+f_{v2})t - \cos(2\pi F_o - f_{v2})t)$$

$$+ \text{terms in } J_1(\beta_1)\ J_1(\beta_2)\,.$$

The terms in $J_1 J_1$ have frequencies like $f_{v1} + f_{v2}$, $f_{v1} - f_{v2}$, $f_{v2} - f_{v1}$ and harmonics.

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